Hessian estimates on Dirichlet and Neumann eigenfunctions of Laplacian

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Outline

- **1. Our focus and motivation**
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1. Our focus and motivation

1. Focus—background

- **Manifold**:
  - (D, g): n-dimensional compact Riemannian manifold with boundary ∂D.
  - ∇ and Δ: the Levi-Civita covariant derivative and the Laplace-Beltrami operator w.r.t. the metric g, respectively.
  - $(\phi, \lambda) \in \mathrm{Eig}(\Delta)$: ϕ is a Dirichlet eigenfunction of $-\Delta$ on D with eigenvalue $\lambda > 0$, i.e. $-\Delta\phi = \lambda\phi$, which is normalized in $L^2(D)$, i.e. $\|\phi\|_{L^2} = 1$.
  - $(\phi, \lambda) \in \mathrm{Eig}_N(\Delta)$: ϕ is a Neumann eigenfunction of $-\Delta$ on D with eigenvalue $\lambda > 0$, i.e. $-\Delta\phi = \lambda\phi$, which is normalized in $L^2(D)$, i.e. $\|\phi\|_{L^2} = 1$.

Focus— backgroud

- **The uniform estimate of** ϕ is that

$$\|\phi\|_\infty \leq c_D \lambda^{\frac{n-1}{4}}$$

for some positive constant c_D.

  - Lars Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218.
  - Daniel Grieser, *Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary*, Comm. Partial Differential Equations **27** (2002), no. 7-8, 1283–1299.

Focus— background

- **The uniform estimate of $\nabla\phi$**: According to [Shi-Xu,2013] and [Hu, Shi and Xui, 2015], there exist two positive constants $c_1(D)$ and $c_2(D)$ such that

$$c_1(D)\sqrt{\lambda}\,\|\phi\|_\infty \leqslant \|\nabla\phi\|_\infty \leqslant c_2(D)\sqrt{\lambda}\,\|\phi\|_\infty, \quad (\phi,\lambda)\in \mathrm{Eig}(\Delta), \tag{1}$$

where we write $\|\nabla\phi\|_\infty := \|\,|\nabla\phi|\,\|_\infty$ for simplicity.

  - Yiqian Shi and Bin Xu, *Gradient estimate of a Dirichlet eigenfunction on a compact manifold with boundary*, Forum Math. **25** (2013), no. 2, 229–240.
  - Jingchen Hu, Yiqian Shi, and Bin Xu, *The gradient estimate of a Neumann eigenfunction on a compact manifold with boundary*, Chin. Ann. Math. Ser. B **36** (2015), no. 6, 991–1000.

Arnaudon, Thalmaier and Wang determined **explicit constants** $c_1(D)$ **and** $c_2(D)$ in (1) for Dirichlet and Neumann eigenfunctions by using martingale method.

- Marc Arnaudon, Anton Thalmaier, and Feng-Yu Wang, *Gradient estimates on Dirichlet and Neumann eigenfunctions*, Int. Math. Res. Not. IMRN (2020), no. 20, 7279–7305.

Steinerberger studied Laplacian eigenfunctions of $-\Delta$ with Dirichlet boundary conditions on bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary and proved a sharp Hessian estimate for the eigenfunctions:

$$\|\mathrm{Hess}\,\phi\|_\infty \lesssim \lambda^{\frac{n+3}{4}} \qquad (\lesssim \lambda\|\phi\|_\infty)$$

where

$$\|\mathrm{Hess}\,\phi\|_\infty := \sup\left\{|\mathrm{Hess}\,\phi(v,v)|(x) : x \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\ |v| = 1\right\}.$$

- Stefan Steinerberger, *Hessian estimates for Laplacian eigenfunctions*, arXiv:2102.02736v1 (2021).

Our question:

For the manifold, how to derive explicit numerical constants $c_1(D)$ and $c_2(D)$ such that

$$c_1(D)\lambda\,\|\phi\|_\infty \leqslant \|\mathrm{Hess}\,\phi\|_\infty \leqslant c_2(D)\lambda\,\|\phi\|_\infty, \quad (\phi,\lambda)\in \mathrm{Eig}(\Delta)? \tag{2}$$

In particular, what is the required curvature assumptions to estimate the constants $c_1(D)$ and $c_2(D)$?

- Note that for eigenfunctions of the Laplacian, one trivially has

$$|\mathrm{Hess}\,\phi| \geq \frac{1}{n}\,|\Delta\phi| = \frac{\lambda}{n}\,|\phi|,$$

and hence there is always the obvious lower bound

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty} \geq \frac{\lambda}{n}.$$

- We shall concentrate in the sequel on upper bounds for

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty}.$$

2. Our work

- $\mathrm{Hess} := \nabla \mathrm{d}$ the Hessian operator on functions.
- Let $\mathrm{Ric}(X, Y) := \nabla^2_{X,Y} - \nabla^2_{Y,X}$ be the Ricci curvature tensor w.r.t. g.
- Let R be the curvature tensor.
- Let $\mathrm{d}^* R(v_1, v_2) := -\mathrm{tr}\nabla_{\cdot} R(\cdot, v_1)v_2$, where

$$\langle \mathrm{d}^* R(v_1, v_2), v_3\rangle = \langle (\nabla_{v_3}\mathrm{Ric}^\sharp)(v_1), v_2\rangle - \langle (\nabla_{v_2}\mathrm{Ric}^\sharp)(v_3), v_1\rangle$$

  for all $v_1, v_2, v_3 \in T_x D$ and $x \in D$.
- Let N be the inward normal unit vector field on ∂D.
- For $X, Y \in T_x\partial D$ and $x \in \partial D$,

$$\mathrm{II}(X, Y) = -\langle \nabla_X N,\ Y\rangle.$$

- For $v_1 \in T_x M$, let $R(v_1) : T_x M \otimes T_x M \to T_x M$ be given by

$$\langle R(v_1)(v_2, v_3), v_4\rangle := \langle \mathrm{R}(v_1, v_2)v_3, v_4\rangle, \quad v_2, v_3, v_4 \in T_x M.$$

- Let

$$|\mathrm{R}|(y) := \sup\left\{\sqrt{\sum_{i,j=1}^{n} \mathrm{R}(e_i, v, w, e_j)^2(y)} : |v| \le 1, |w| \le 1, v, w \in T_y D\right\}$$

for an orthonormal base $\{e_i\}_{i=1}^n$ of $T_y D$.

Theorem 1 [Ch.-Thalmaier-Wang, 2023]

Let D be an n-dimensional complete Riemannian manifold without boundary. Assume that there exist constants K_0, K_1, K_2 such that $\mathrm{Ric} \ge -K_0$, $|\mathrm{R}| \le K_1$ and $|\mathrm{d}^* R + \nabla \mathrm{Ric}| \le K_2$. Then

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty} \le \left(K_1 \sqrt{\frac{2}{2K_0^+ + \lambda}} + \frac{K_2}{2K_0^+ + \lambda}\right)\mathrm{e} + (\lambda + 2K_0^+)\mathrm{e}.$$

Theorem 2 [Ch.-Thalmaier-Wang, 2023]

Let D be an n-dimensional compact Riemannian manifold with boundary ∂D. Suppose that $|\mathrm{Ric}| \le K_0$, $|\mathrm{R}| \le K_1$ and $|\mathrm{d}^* R + \nabla \mathrm{Ric}| \le K_2$ on D, and that $|\nabla^2 N| \le \beta$ on D, $|\mathrm{II}| \le \sigma$ on the boundary ∂D. Let $\alpha \in \mathbb{R}$ be such that

$$\frac{1}{2}\Delta \rho_{\partial D} \le \alpha. \tag{3}$$

Then for non-trivial $(\phi, \lambda) \in \mathrm{Eig}(\Delta)$,

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty} \le \left(C_\lambda(D) \wedge \tilde{C}_\lambda(D)\right)\lambda;$$

denote by λ_1 the first Neumann eigenvalue of $-\Delta$, then

$$\frac{\|\mathrm{Hess}\phi\|_\infty}{\|\phi\|_\infty} \le \left(C_{\lambda_1}(D) \wedge \tilde{C}_{\lambda_1}(D)\right)\lambda,$$

where

$$C_\lambda(D) := \frac{\mathrm{e}}{\lambda}\sqrt{\frac{\lambda+K_0}{2}}\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+K_0)}\right) + \frac{\sigma(n-1)\sqrt{\mathrm{e}}}{\lambda}\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+2K_0)}\right)$$
$$+ \frac{\mathrm{e}}{\lambda(\lambda+K_0)}\left((3n+12)\beta + 2K_0 + \frac{(n-1)\beta}{\sqrt{\lambda+K_0}}\right)\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+K_0)}\right)\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+2K_0)}\right)$$
$$+ \frac{\mathrm{e}}{\lambda}\left(2\alpha^+ + \sqrt{\frac{2(\lambda+K_0)}{\pi}} + \frac{\sqrt{\pi}(\lambda+K_0)}{4\left(2\sqrt{\pi}\alpha^+ + \sqrt{2(\lambda+K_0)}\right)}\right)\left(\frac{2K_1\sqrt{\lambda+2K_0}+K_2}{2(\lambda+2K_0)} + \sqrt{\lambda+2K_0}\right),$$

and

$$
\begin{aligned}
\tilde{C}_\lambda(D) :=\ & \frac{\mathrm{e}}{\lambda}\sqrt{\frac{\lambda+K_0}{2}}\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+K_0)}\right) + \frac{\sigma(n-1)\sqrt{\mathrm{e}}}{\lambda}\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+2K_0)}\right) \\
& + \frac{\mathrm{e}}{\lambda(\lambda+K_0)}\left((3n+12)\beta + 2K_0 + \frac{(n-1)\beta}{\sqrt{\lambda+K_0}}\right)\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+K_0)}\right)\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+2K_0)}\right) \\
& + \frac{\sqrt{\mathrm{e}}}{\lambda}\left(K_1 + \frac{K_2}{2\sqrt{\lambda+2K_0}}\right)\sqrt{\left(\frac{2\alpha^+}{\sqrt{\lambda+2K_0}} + \sqrt{\frac{2}{\pi}}\right)^2 + 1} \\
& + \frac{\sqrt{\mathrm{e}}}{\lambda}\left(2\alpha^+\sqrt{\lambda+2K_0} + \sqrt{\frac{2}{\pi}(\lambda+2K_0)}\right)\mathbb{1}_{\left\{\alpha^+ > \left(2-\sqrt{\frac{1}{2\pi}}\right)\sqrt{\lambda+2K_0}\right\}} \\
& + \frac{\sqrt{\mathrm{e}}}{\lambda}\left(2(\lambda+2K_0) + \frac{1}{8}\left(2\alpha^+ + \sqrt{\frac{2}{\pi}(\lambda+2K_0)}\right)^2\right)\mathbb{1}_{\left\{\alpha^+ \le \left(2-\sqrt{\frac{1}{2\pi}}\right)\sqrt{\lambda+2K_0}\right\}}.
\end{aligned}
$$

Remark

With constants $K_0, \theta > 0$ such that $\mathrm{Ric} \geq -K_0$ on D and $H \geq -\theta$ on the boundary ∂D, where $H(x)$ is the mean curvature of D at $x \in D$, let

$$\alpha = \frac{1}{2} \max\left\{\theta, \sqrt{(n-1)K_0}\right\}.$$

Then estimate (3) holds for such α.

Theorem 3 [Ch.-Thalmaier-Wang, 2023]

Let D be an n-dimensional compact Riemannian manifold with boundary ∂D. Assume that $\mathrm{Ric} \geq -K_0$, $|\mathrm{R}| \leq K_1$ and $|\mathrm{d}^* R + \nabla \mathrm{Ric}| \leq K_2$ on D, and that $\mathrm{II} \geq -\sigma_1$ and $|\nabla^2 N - R(N)| \leq \sigma_2$ on the boundary ∂D. For $h \in C^\infty(D)$ with $\min_D h = 1$ and $N \log h|_{\partial D} \geq 1$, let $K_{h,\alpha} := \sup_D\{-\Delta \log h + \alpha |\nabla \log h|^2\}$ with α a positive constant. Then, for any non-trivial $(\phi, \lambda) \in \mathrm{Eig}_N(\Delta)$,

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty} \leq C_{N,\lambda}(D)\lambda;$$

denoting by λ_1 the first Neumann eigenvalue of $-\Delta$, then

$$\frac{\|\mathrm{Hess}\phi\|_\infty}{\|\phi\|_\infty} \leq C_{N,\lambda_1}(D)\,\lambda,$$

where

$$C_{N,\lambda}(D) = e\left(1 + \frac{K_1 + 2K_0^+ + (2\sigma_1^+ + \delta)K_{h,2(\sigma_1^+ + \delta)}}{\lambda} + \frac{K_2}{\lambda\sqrt{2\lambda + 4K_0^+ + (4\sigma_1^+ + 2\delta)K_{h,2(\sigma_1^+ + \delta)}}}\right)\|h\|_\infty^{2\sigma_1^+}$$

$$+ \frac{\sigma_2 e}{2(\sigma_1^+ + \delta)\lambda}\|h\|_\infty^{2\sigma_1^+ + \delta}\sqrt{2\lambda + 4K_0^+ + (4\sigma_1^+ + 2\delta)K_{h,(2\sigma_1^+ + \delta)}}$$

for any $\delta > 0$ ($\delta \geq 0$ if $\sigma_1^+ > 0$).

Condition (H)

There exists a non-negative constant θ such that $\mathrm{II} \le \theta$ and a positive constant r_0 such that on $\partial_{r_0} D := \{x \in D : \rho_\partial(x) \le r_0\}$ the distance function ρ_∂ to the boundary ∂D is smooth and there exists some constant k such that $\mathrm{Sect} \le k$ on $\partial_{r_0} D$.

- Feng-Yu Wang, *Estimates of the first Neumann eigenvalue and the log-Sobolev constant on non-convex manifolds*, Math. Nachr. **280** (2007), no. 12, 1431–1439.

Under Condition **(H)**, F.-Y. Wang (see Theorem 3.2.9 in [Wang, 2007]) construct h:

$$\log h(x) = \frac{1}{\Lambda_0} \int_0^{\rho_\partial(x)} (\ell(s) - \ell(r_1))^{1-n} \, ds \int_{s \wedge r_1}^{r_1} (\ell(u) - \ell(r_1))^{n-1} \, du$$

where

$$\ell(t) := \begin{cases} \cos \sqrt{k}t - \frac{\theta}{\sqrt{k}} \sin \sqrt{k}t, & k > 0, \\ 1 - \theta t, & k = 0, \\ \cosh \sqrt{-k}t - \frac{\theta}{\sqrt{-k}} \sinh \sqrt{-k}t, & k < 0, \end{cases} \tag{4}$$

$r_1 := r_0 \wedge \ell^{-1}(0)$ and

$$\Lambda_0 := (1 - \ell(r_1))^{1-n} \int_0^{r_1} (\ell(s) - \ell(r_1))^{n-1} \, ds.$$

2. Construction of h

Corollary 4 [Ch.-Thalmaier-Wang, 2023]

Let D be a compact n-dimensional Riemannian manifold with boundary ∂D. Assume that $\mathrm{Ric} \geq -K_0$, $|R| \leq K_1$ and $|\mathrm{d}^*R + \nabla \mathrm{Ric}| \leq K_2$ on D, and that $\mathrm{II} \geq -\sigma_1$, and $|\nabla^2 N - R(N)| \leq \sigma_2$ on the boundary ∂D for $\sigma_1, \sigma_2 \geq 0$. Assume that Condition $(\mathbf{H})$ is satisfied. Then the Hessian estimate of Neumann eigenfunctions in Theorem 3 remains valid under replacing

$$\sigma_1,\ K_{h,\alpha} \text{ and } \|h\|_\infty$$

by

$$\sigma^+,\quad K_\alpha := \frac{n}{r_1} + \alpha \text{ and } \mathrm{e}^{nr_1/2}$$

respectively.

3. Sketch of proofs

3. Notations

- Let $X_\cdot^x$ be a Brownian motion for each $x \in M$.
- For $f \in C_b(M)$, $P_t f(x) = \mathbb{E}[f(X_t^x)], \quad t \geq 0.$
- The damped parallel transport $Q_t : T_xM \to T_{X_t}M$ is defined as the solution, along the paths of X_t, to the covariant ordinary differential equation

$$\mathrm{D}Q_t = -\mathrm{Ric}^\sharp Q_t\, dt, \quad Q_0 = \mathrm{id}_{T_xM}, \tag{5}$$

where $\mathrm{D}Q_t = //_t\, d\, //_t^{-1} Q_t$ and $//_t$ is the parallel transport along the paths of X_t.

If the manifold has no boundary and $\mathrm{Ric} \geq -K_0$ for some constant $K_0 \geq 0$, then

- one has the Bismut-type formula

$$\nabla P_t f(x) = \mathbb{E}\left[f(X_t(x)) \int_0^t \langle Q_t(\dot{k}(s)v), //_s\, dB_s\rangle\right],$$

  where $k \in C_b^1([0,\infty), \mathbb{R})$ satisfying $k(0) = 1$ and $k(s) = 0$ for $s \geq t$;
- taking $f = \phi$, and using $P_t\phi = e^{-\frac{1}{2}\lambda t}\phi$ yields the upper bound of $\|\nabla\phi\|_\infty$.

For the Neumann boundary, the idea is also to use the Bismut type formula for Neumann semigroup.

Suppose the manifold D has boundary and $(\phi, \lambda) \in \mathrm{Eig}(\Delta)$.

Step 1 For $v \in T_xM$ and any $k \in C_b^1([0,\infty);\mathbb{R})$ such that $k(0) = 1$ and $k(s) = 0$ for $s \geq T$, i.e., k bounded with bounded derivative, the process

$$k(t)\mathrm{e}^{\lambda t/2}\langle \nabla\phi(X_t), Q_t(v)\rangle - \mathrm{e}^{\lambda t/2}\phi(X_t)\int_0^t \langle \dot{k}(s)Q_s(v), //_s dB_s\rangle, \quad t \leq \tau_D$$

is a martingale, where $\tau_D = \inf\{t > 0 : X_t(x) \in \partial D\}$.

Step 2 By taking expectation at time $t = 0$ and $t = T \wedge \tau_D$,

$$\begin{aligned}\langle \nabla\phi, v\rangle &= \mathbb{E}\left[k(T\wedge\tau_D)e^{\lambda(T\wedge\tau_D)/2}\langle\nabla\phi(X_{T\wedge\tau_D}), Q_{T\wedge\tau_D}(v)\rangle\right] \\ &\quad - \mathbb{E}\left[\phi(T\wedge\tau_D)e^{\lambda(T\wedge\tau_D)/2}\int_0^{T\wedge\tau_D}\langle\dot{k}(s)Q_s v, //_s dB_s\rangle\right] \\ &= \mathbb{E}\left[1_{\{T\geq\tau_D\}}e^{\lambda\tau_D/2}\langle\nabla\phi(X_{\tau_D}), Q_{\tau_D}(v)\rangle\right] \\ &\quad - \mathbb{E}\left[\phi(T\wedge\tau_D)e^{\lambda(T\wedge\tau_D)/2}\int_0^{T\wedge\tau_D}\langle\dot{k}(s)Q_s v, //_s dB_s\rangle\right].\end{aligned}$$

- $|\nabla\phi|(x) = |N(\phi)|(x), \quad x \in \partial D.$
- $k(s) = \frac{t-s}{t}, \quad s \in [0, t].$

3.1. Case I: no boundary

Operator-valued process W_t

For $w \in T_xM$ define an operator-valued process $W_t(\cdot, w) : T_xM \to T_{X_t}M$ by

$$W_t(\cdot, w) = Q_t \int_0^t Q_r^{-1} R(//_r\, dB_r, Q_r(\cdot))Q_r(w) - Q_t \int_0^t Q_r^{-1}(\nabla \mathrm{Ric}^\sharp + d^*R)(Q_r(\cdot), Q_r(w))\, dr.$$

Then the process $W_t(\cdot, w)$ is the solution to the covariant Itô equation

$$\begin{cases} \mathrm{D}W_t(\cdot, w) = R(//_t dB_t, Q_t(\cdot))Q_t(w) - (d^*R + \nabla \mathrm{Ric}^\sharp)(Q_t(\cdot), Q_t(w))\, dt \\ \qquad\qquad - \mathrm{Ric}^\sharp(W_t(\cdot, w))\, dt, \\ W_0(\cdot, w) = 0. \end{cases}$$

Bismut-type Hessian formula

Theorem ([Elworthy-Li, 1998])

Assume k, ℓ are bounded adapted processes with paths in the Cameron-Martin space $L^{1,2}([0,T];[0,1])$ such that for $0 < S < T$,

- $k(0) = 1$ and $k(s) = 0$ for $s \geq S$;
- $\ell(s) = 1$ for $s \leq S$ and $\ell(s) = 0$ for $s \geq T$.

Then for $f \in \mathcal{B}_b(M)$, we have

$$(\mathrm{Hess}_x P_T f)(v,v) = -\mathbb{E}^x\left[f(X_T)\int_0^T \langle W_s(\dot{k}(s)v, v), //_s dB_s\rangle\right] + \mathbb{E}^x\left[f(X_T)\int_S^T \langle Q_s(\dot{\ell}(s)v), //_s dB_s\rangle \int_0^S \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\right]. \quad (6)$$

- K. David Elworthy and Xue-Mei Li, Bismut type formulae for differential forms, C. R. Acad. Sci. Paris Sér. I Math. 327 (1998), no. 1, 87-92.

Bismut-type Hessian formula–history

- Localization of Elworthy-Li's formula by [Arnaudon-Plank-Thalmaier, 2003].
- The study of the Hessian of a Feynman-Kac semigroup has been pushed forward by [Li, 2016], [Li, 2018] and [Thompson, 2019].
- **An intrinsic formula for** $\mathrm{Hess} P_t f$ with only one test function has been given in [Stroock,1996] for a compact Riemannian manifold by the path theory.
- **Martingale method** is used to extend the intrinsic formula of $\mathrm{Hess} P_t f$ by [Chen-Ch. -Thalmaier, 2021].

Theorem 4 [Chen-Ch.-Thalmaier, SPDE, 2022]

Let D be a compact and complete manifold without boundary. For $k \in C_b^1([0,\infty);\mathbb{R})$ with $k(0)=1$ and $k(t)=0$ for $t \geq T$, one has for $v \in T_xM$,

$$(\mathrm{Hess}\, P_T f)(v,v) = -\mathbb{E}^x\left[f(X_T)\int_0^T \langle W_s^k(\dot{k}(s)v, v), //_s dB_s\rangle\right] + \mathbb{E}\left[f(X_T(x))\left(\left(\int_0^T \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\right)^2 - \int_0^T |Q_s(\dot{k}(s)v)|^2\, ds\right)\right], \quad (7)$$

where

$$W_t^k(w,v) = Q_t \int_0^t Q_r^{-1} R(//_r\, dB_r, Q_r(w)) Q_r(k(r)v) - Q_t \int_0^t Q_r^{-1}(\nabla \mathrm{Ric}^\sharp + \mathrm{d}^* R)(Q_r(w), Q_r(k(r)v))\, dr.$$

- If $\mathrm{Ric} \geq -K_0$, then $|Q_t| \leq \mathrm{e}^{\frac{K_0 t}{2}}$;
- An appropriate estimate of $\mathbb{E} \int_0^t |W_t^k(v, v)|^2 \, \mathrm{d}t$;
- If $f = \phi$ and $(\phi, \lambda) \in \mathrm{Eig}(\Delta)$, then

$$\|\mathrm{Hess}\, \phi\|_\infty \leq \left(K_1 \sqrt{t} + \frac{K_2 t}{2}\right) \mathrm{e}^{(K_0^+ + \lambda/2)t} \|\phi\|_\infty + \frac{2\mathrm{e}^{(K_0^+ + \lambda/2)t}}{t} \|\phi\|_\infty$$

for any $t > 0$.

- Letting $t = \frac{1}{K_0^+ + \lambda/2}$ then yields the estimate in Theorem 1.

3.2. Case II: Dirichlet boundary

Two Methods

- **Method 1:**
  - construct a martingale to connect $\mathrm{Hess}\phi$ and $\nabla\phi$;
  - via estimating the boundary value of $\|\mathrm{Hess}\,\phi\|_{\partial D,\infty}$ to give the estimate

$$\|\mathrm{Hess}\,\phi\|_\infty \le (\cdots)\|\nabla\phi\|_\infty \overset{\text{Arnaudon-Thalmaier-Wang's result}}{\le} (\cdots)\|\phi\|_\infty.$$

- **Method 2:**
  - construct a martingale to connect $\mathrm{Hess}\,\phi$ and ϕ;
  - via estimating the boundary value of $\|\mathrm{Hess}\,\phi\|_{\partial D,\infty}$ and $\|\nabla\,\phi\|_{\partial D,\infty}$ to give the estimate

$$\|\mathrm{Hess}\,\phi\|_\infty \le (\cdots)\|\phi\|_\infty.$$

First type of martingale

The process

$$M_t := \mathrm{e}^{\lambda t/2} \mathrm{Hess}\,\phi(Q_t(k(t)v), Q_t(v)) + \mathrm{e}^{\lambda t/2} \mathbf{d}\phi(W_t^k(v,v)) - \mathrm{e}^{\lambda t/2} \mathbf{d}\phi(Q_t(v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s \rangle \tag{8}$$

is a martingale on $[0, \tau_D]$.

-
$$\|\mathrm{Hess}\,\phi\|_\infty \le C_1 \|\mathrm{Hess}\,\phi\|_{\partial D, \infty} + C_2 \|\nabla \phi\|_\infty . \tag{9}$$

Second type of martingale

The process

$$
\begin{aligned}
N_t :=& e^{\lambda t/2}\mathrm{Hess}\phi(Q_t(k(t)v), Q_t(k(t)v)) + e^{\lambda t/2}\mathbf{d}\phi(W_t^k(v,\ k(t)v)) \\
& - 2e^{\lambda t/2}\mathbf{d}\phi(Q_t(k(t)v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle \\
& - e^{\lambda t/2}\phi(X_t) \int_0^t \langle W_s^k(v, \dot{k}(s)v), //_s dB_s\rangle \\
& + e^{\lambda t/2}\phi(X_t)\left(\left(\int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\right)^2 - \int_0^t |Q_s(\dot{k}(s)v)|^2\, ds\right) \qquad (10)
\end{aligned}
$$

is a martingale on $[0, \tau_D]$.

- $$\|\mathrm{Hess}\,\phi\|_\infty \le C_1\|\mathrm{Hess}\,\phi\|_{\partial D,\infty} + C_2\|\nabla\,\phi\|_{\partial D,\infty} + C_3\|\phi\|_\infty. \qquad (11)$$

Hessian estimate of the boundary value of $\mathrm{Hess}\,\phi$

Lemma 5

Assume $|\mathrm{II}| \le \sigma$ and $H \ge -\theta$ on the boundary ∂D and $|\mathrm{Ric}| \le K_0$. Assume that N is the extended vector field of the normal vector field from ∂D to the whole D and $\beta := \|\nabla^2 N\|_\infty$. Letting $\alpha \in \mathbb{R}$ be such that $\frac{1}{2}\Delta\rho_{\partial D} \le \alpha$, then for $x \in \partial D$,

$$\left|\mathrm{Hess}(\phi)\right|(x) \le \sqrt{\frac{(\lambda + K_0)\mathrm{e}}{2}} f\left(\frac{1}{\lambda + K_0}, \alpha\right) \|\phi\|_\infty + \left\{ \sqrt{\mathrm{e}}\left[\frac{(3n+12)\beta + 2K_0}{\lambda + K_0} + \frac{(n-1)\beta}{(\lambda + K_0)^{3/2}}\right] f\left(\frac{1}{\lambda + K_0}, \alpha\right) + \sigma(n-1)\right\} \|N(\phi)\|_{\partial D, \infty},$$

where

$$f(t, \alpha) = \alpha^+ + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - \mathrm{e}^{-\alpha^2 s/2}}{\sqrt{2\pi s^3}}\, ds.$$

Hessian estimate of the boundary value of $\mathrm{Hess}\,\phi$

Given $x \in \partial D$, let $\{X_i\}_{1\le i\le n}$ be an orthonormal basis of T_xD with $X_1 = N$.

- If $X_i, X_j \in T_x\partial D$, i.e. $i, j \neq 1$, then $\langle \nabla\phi, X_j\rangle|_{\partial D} = 0$ and

$$|\mathrm{Hess}(\phi)(X_i, X_j)| = |-N(\phi)\langle N, \nabla_{X_i}X_j\rangle| \le \sigma|N(\phi)|.$$

- If $X_i = X_j = N$, i.e. $i = j = 1$, then $\nabla_N N|_{\partial D} = 0$ and

$$|\mathrm{Hess}(\phi)(N, N)| = |N^2(\phi)| \le (n-1)\sigma|N(\phi)|.$$

- If $X_j \in T_x\partial D$ and $X_i = N$ (i.e. $j \neq 1$ and $i = 1$), then

$$|\mathrm{Hess}(\phi)(X_j, N)|(x) = |NX_j(\phi)|(x).$$

Estimate of $\|NX_j(\phi)\|_{\partial D,\infty}$

- Fix $x \in \partial D$. For small $\varepsilon > 0$, let $x^\varepsilon = \exp_x(\varepsilon N)$, where N is the inward normal vector field of ∂D such that $\nabla N(x) = 0$. We extend X_j and N to the inner of D. Since $\phi|_{\partial D} = 0$, it is easy to see that $X_j(\phi)|_{\partial D} = 0$ and then

$$
\begin{aligned}
|NX_j(\phi)|(x) &= \lim_{\varepsilon\to 0} \frac{|X_j(\phi)(x^\varepsilon)|}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{|\langle \nabla\phi, X_j\rangle(x^\varepsilon)|}{\varepsilon} \\
&\le e^{\frac{\lambda t}{2}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big|\mathbb{E}^{x^\varepsilon}[\mathbb{1}_{\{\tau_D<t\}} k(\tau_D)\langle \nabla\phi(X^\varepsilon_{\tau_D}), Q_{\tau_D}(X_j)\rangle] \\
&\quad - \mathbb{E}^{x^\varepsilon}\Big[\mathbb{1}_{\{t\le\tau_D\}}\phi(X^\varepsilon_t)\int_0^{t\wedge\tau_D} \dot{k}(s)\langle Q_s(X_j), //_s dB_s\rangle\Big]\Big|,
\end{aligned}
$$

where

$$
k(t)e^{\frac{\lambda t}{2}}\langle \nabla\phi(X^\varepsilon_t), Q_t(X_j)\rangle - e^{\frac{\lambda t}{2}}\phi(X^\varepsilon_t)\int_0^t \langle \dot{k}(s)Q_s(X_j), //_s dB_s\rangle
$$

is a martingale and $k \in C^1([0,t];[0,1])$ such that $k(0) = 1$ and $k(t) = 0$.

Gradient estimate of the boundary value of $N(\phi)$

Lemma 6

Let $\alpha \in \mathbb{R}$ such that

$$\frac{1}{2}\Delta\rho_{\partial D} \leq \alpha.$$

Then for any $t > 0$,

$$\|\nabla\phi\|_{\partial D,\infty} = \|N(\phi)\|_{\partial D,\infty} \leq \|\phi\|_\infty \mathrm{e}^{\lambda t/2} f(t,\alpha)$$

where

$$f(t,\alpha) = \alpha^+ + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1-\mathrm{e}^{-\alpha^2 s/2}}{\sqrt{2\pi s^3}}\, ds.$$

1. Main references

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Thank you!